

# Large phase shift of (1+1)-dimensional nonlocal spatial solitons in lead glass

Qian Shou,<sup>1</sup> Miao Wu,<sup>1</sup> and Qi Guo<sup>1,\*</sup>

<sup>1</sup>*Laboratory of Nanophotonic Functional Materials and Devices,  
South China Normal University, Guangzhou 510006*

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The large phase shift of strongly nonlocal spatial optical soliton(SNSOS) in the (1+1)-dimensional [(1+1)D] lead glass is investigated using the perturbation method. The fundamental soliton solution of the nonlocal nonlinear Schrödinger equation(NNLSE) under the second approximation in strongly nonlocal case is obtained. It is found that the phase shift rate along the propagation direction of such soliton is proportional to the degree of nonlocality, which indicates that one can realize  $\pi$ -phase-shift within one Rayleigh distance in (1+1)D lead glass. A full comprehension of the nonlocality-enhancement to the phase shift rate of SNSOS is reached via quantitative comparisons of phase shift rates in different nonlocal systems.

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## 1. Introduction

Nonlocal spatial solitons have been the subject of intensive experimental and theoretical work [1–4] since the pioneering work done by Snyder and Mitchell [5]. The most prominent innovation in their work is that they transforms the complex nonlocal nonlinear Schrödinger equation (NNLSE) into a simple case of linear propagation of light in a quadratic self-induced index well [5]. Nonlocal nonlinearity is typically the result of certain transport processes, such as the charge drift in photorefractive crystals [6] and the heat transfer in thermal nonlinear media [2] or, long-range interaction, such as the molecular reorientations in liquid crystals [7]. Due to the nature of the nonlocality, solitons in nonlocal nonlinear media exhibit several distinct properties that are not possible in local settings. This includes, on one hand, resulting from the spatial ‘averaging’ character of the nonlocality, the arrest of catastrophic collapse [3], the ability to support the formation of complex optical spatial solitons, such as higher-order solitons [8, 9] and vortex solitons [2, 10]. On the other hand, out-of-phase solitons attraction [11, 12], long-range interactions between solitons [13] as well as the solitons and the boundaries [14–16] in strongly nonlocal media have also been carried out or predicted due to the fact that the interactions are mediated by the light-induced refractive index which is ‘enlarged’ by the nonlocal response.

Except for the ‘averaging’ and the ‘enlarging’ features of the nonlocality, there exists an ‘enhancing’ effect of

the nonlocality on the phase shift of the SNSOS. Although very large in fact, the phase shift of SNSOS is considered a trivial term, for a long time, and is neglected by the Snyder-Mitchell (SM) model [5]. The first work focused on the phase shift of SNSOS was done by Guo *et al.* [17, 18]. They predicted a large phase shift rate of SNSOS, which is  $\alpha^2$  times ( $\alpha$  is the degree of nonlocality defined as  $\alpha = w_m/\mu$  where  $w_m$  is the characteristic length of the response function and  $\mu$  is the beam width), explicitly 100 times for the lower limit of the strongly nonlocality, larger than that of the local counterpart. Guo’s conclusion results from a strongly nonlocal (SN) model in which the large phase shift is included having a dominating term proportional to the soliton critical power.

SN model can rigorously transform to SM model with a function transformation involving large phase shift term [19]. Both of them are derived from a phenomenological and regular (or at least twice-differentiable at  $\mathbf{r} = 0$ ) response function  $R(\mathbf{r})$ . In the nematic liquid crystal (NLC) and lead glass(LG), the two media found so far in which SNSOSs can form, the response functions are singular at every source point (irregular) and therefore one can not obtain accurate solution of NNLSE based on SN model and SM model even in the strongly nonlocal case[19]. Ouyang *et al.* took the higher order (the forth and the sixth) terms of the light induced refractive index as the perturbation to the quadratic index well and obtained considerably accurate analytical soliton solutions in (1+1)D [20] and (1+2)D [21] NLC. The perturbation solution are different from Gaussian-type solution given by SN model[19], but still indicated

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\* Corresponding author: guoq@scnu.edu.cn

nonlocality-enhanced large phase shifts of SNSOSs. The first theoretical and experimental study focused on the SNSOS phase shift was carried out in (1+2)D cylindrical LG by Shou *et al.* [22]. They retained the terms of the Taylor expansion of the light-induced refractive index up to the second order whose coefficient is the on-axis light intensity. The phase shift rate in (1+2)D LG was predicted to be much smaller than the result based on SN model, but is still more than one order larger than that in the local media. More meaningful, Shou *et al.* observed a linear modulation of the soliton power on the phase shift of the SNSOS [22], which coincides with Guo's prediction, indicating that the nonlocality enhancement to the phase shift of SNSOS stems from the fact that the light-induced refractive index, which directly contributes to the phase shift, is induced not by the light intensity but by the power of the whole beam.

In this paper, we investigate the phase shift of SNSOS in (1+1)D LG in the formalism of perturbation theory. The perturbation solution of the fundamental soliton is obtained under the second approximation. The result indicates that the phase shift of SNSOS in (1+1)D LG is proportional to the degree of nonlocality which is at least one order larger than the result for the local solitons. It will also be shown how the degree of nonlocality affects, or explicitly speaking, enhances the phase shift rate in different nonlocal systems.

## 2. The fundamental strongly nonlocal soliton solution under the second approximation

We consider a (1+1)D LG with thermal nonlocal nonlinear response occupying the region  $-L \leq x \leq L$ . The propagation behavior of a light beam  $u$  propagating along the  $z$  axis is governed by the NNLSE, coupled to the Poisson equation describing the light-induced nonlinear refractive index variation  $N$ ,

$$i\frac{\partial u}{\partial z} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + Nu = 0, \quad (1)$$

$$\frac{d^2 N}{dx^2} = -|u|^2. \quad (2)$$

The nonlocal response function in (1+1)D LG under the first-kind boundary condition  $N(\pm L) = 0$  can be given as [23]

$$G(x, \xi) = \begin{cases} \frac{(x+L)(\xi-L)}{2L}, & (x \leq \xi) \\ \frac{(\xi+L)(x-L)}{2L}, & (\xi \leq x) \end{cases} \quad (3)$$

According to the Green function method, the nonlinear refractive index in LG can be written in the form of

$$N(x) = -\int_{-L}^L G(x, \xi)|u(\xi, z)|^2 d\xi. \quad (4)$$

It is obvious that the response function in Eq.(3) is not differentiable at the source point  $x = \xi$  (irregular) and

therefore cannot be dealt with SN model [17]. We use the perturbation method, previously extended to solve the NNLSE by Ouyang *et al.* [20, 21], to calculate the fundamental soliton solution of the NNLSE. For the soliton state  $u(x, z)$ , we have  $|u(-x, z)|^2 = |u(x, z)|^2$  and  $u(x, z) = u(x, 0)$ . On the analogy of the potential in quantum mechanics which determines the state of the particle movement, we define the nonlinearity-induced trapping 'potential', explicitly the light-induced refractive index, which can determine the beam propagation behavior,

$$V(x) = \int_{-L}^L G(x, \xi)|u(\xi, z)|^2 d\xi. \quad (5)$$

Then Eq.(1) can be reduced to

$$i\frac{\partial u}{\partial z} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - V(x)u = 0. \quad (6)$$

Taking the Taylor's expansion of  $V(x)$  at  $x = 0$ , we obtain

$$V(x) = V_0 + \frac{1}{2\mu^4}x^2 + \alpha x^4 + \beta x^6 + \dots, \quad (7)$$

where

$$V_0 = V(0), \quad (8a)$$

$$\frac{1}{\mu^4} = V^{(2)}(0), \quad (8b)$$

$$\alpha = \frac{1}{4!}V^{(4)}(0), \quad (8c)$$

$$\beta = \frac{1}{6!}V^{(6)}(0). \quad (8d)$$

In the strongly nonlocal case,  $V(x)$  is effective mainly within the beam region. Consequently the terms  $\alpha x^4$  and  $\beta x^6$  are, respectively, one and two orders of magnitude smaller than the term  $x^2/(2\mu^4)$  [20] and then can be viewed as the perturbations. By substituting Eq.(7) into Eq.(6) and neglecting the higher-order terms, we obtain

$$i\frac{\partial u}{\partial z} = \left[ -\frac{1}{2}\frac{\partial^2}{\partial x^2} + V_0 + \frac{1}{2\mu^4}x^2 + \alpha x^4 + \beta x^6 \right] u. \quad (9)$$

Taking a transformation

$$u(x, z) = \phi(x) \exp[-i(\varepsilon + V_0)z], \quad (10)$$

we arrive at

$$\left[ -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2\mu^4}x^2 + \alpha x^4 + \beta x^6 \right] \phi = \varepsilon \phi. \quad (11)$$

If  $\alpha = 0$  and  $\beta = 0$ , Eq.(11) reduces to the well-known stationary Schrödinger equation for a harmonic oscillator. Following the perturbation method we obtain the fundamental soliton solution under the second approximation

$$\begin{aligned} \phi_0(A, \alpha, \beta, x) \approx & A \left( \frac{1}{\pi\mu^2} \right)^{1/4} \exp \left( -\frac{x^2}{2\mu^2} \right) \\ & \times \left[ 1 + \alpha \left( \frac{9\mu^6}{16} - \frac{3\mu^4}{4}x^2 - \frac{\mu^2}{4}x^4 \right) \right. \\ & + \alpha^2 \left( -\frac{1247\mu^{12}}{512} + \frac{141\mu^{10}}{64}x^2 + \frac{53\mu^8}{64}x^4 + \frac{13\mu^6}{48}x^6 + \frac{\mu^4}{32}x^8 \right) \\ & \left. + \beta \left( \frac{55\mu^8}{32} - \frac{15\mu^6}{8}x^2 - \frac{5\mu^4}{8}x^4 - \frac{\mu^2}{6}x^6 \right) \right], \end{aligned} \quad (12)$$

and

$$\varepsilon_0 \approx \frac{1}{2\mu^2} + \frac{3\mu^4\alpha}{4} - \frac{21\mu^{10}\alpha^2}{8} + \frac{15\mu^6\beta}{8}. \quad (13)$$

In the strongly nonlocal case,  $\alpha$  and  $\beta$  are very small, and accordingly, so is the difference between the fundamental soliton solution under the second approximation  $\phi_0(A, \alpha, \beta, x)$  and that under the zeroth approximation  $\phi_0(A, 0, 0, x)$ .  $V(x)$  can be approximately given by

$$\begin{aligned} V(x) \approx & \int_{-L}^L G(x, \xi) \phi_0^2(A, 0, 0, \xi) d\xi \\ = & \frac{A^2}{2} \left\{ \frac{\mu}{\pi} \left[ \exp \left( -\frac{x^2}{\mu^2} \right) - \exp \left( -\frac{L^2}{\mu^2} \right) \right] \right. \\ & \left. - \text{Lerf} \left( \frac{L}{\mu} \right) + x \text{erf} \left( \frac{x}{\mu} \right) \right\}, \end{aligned} \quad (14)$$

where

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2} d\xi. \quad (15)$$

Combining Eq.(8), we have

$$A^2 \approx \frac{\sqrt{\pi}}{\mu^3}, \quad (16a)$$

$$V_0 \approx \frac{\sqrt{\pi}}{2\mu^3} \left\{ \frac{\mu}{\sqrt{\pi}} \left[ 1 - \exp \left( -\frac{L^2}{\mu^2} \right) \right] - \text{Lerf} \left( \frac{L}{\mu} \right) \right\}, \quad (16b)$$

$$\alpha \approx -\frac{1}{12\mu^6}, \quad (16c)$$

$$\beta \approx \frac{1}{60\mu^8}. \quad (16d)$$

Inserting Eq.(12) into Eq.(10), we find the fundamental soliton solution in (1+1)D LG,

$$\begin{aligned} u(x, z) \approx & A \left( \frac{1}{\pi\mu^2} \right)^{1/4} \exp \left( -\frac{x^2}{2\mu^2} \right) \exp(i\gamma z) \\ & \times \left[ 0.9649 + a \frac{x^2}{\mu^2} + b \frac{x^4}{\mu^4} + c \frac{x^6}{\mu^6} + 0.0002 \frac{x^8}{\mu^8} \right], \end{aligned} \quad (17)$$

where  $A^2 \approx \frac{\sqrt{\pi}}{\mu^3}$ ,  $a = 0.0386$ ,  $b = 0.0162$ ,  $c = -0.0009$ ,  $d = 0.0002$ , and the phase shift rate is of the form

$$\gamma = -V_0 - \varepsilon_0 \approx \frac{1}{2\mu^2} \left[ \frac{\sqrt{\pi}L}{\mu} \text{erf} \left( \frac{L}{\mu} \right) + \exp \left( -\frac{L^2}{\mu^2} \right) - 1.87 \right]. \quad (18)$$

It is important to notice that in Eq.(17),  $\mu$ , defined in Eq. (8b), is visualized as the beam width. The power of the soliton is approximatively given by

$$P = \int_{-\infty}^{+\infty} |u(x, z)|^2 dx \approx A^2 \approx \frac{\sqrt{\pi}}{\mu^3}. \quad (19)$$

In the above equations,  $L$  plays the part of the characteristic length  $w_m$  of the response function, since  $w_m$  of the LG modeled by Eq.(2) is intrinsically infinite but cut off by its boundary [15]. Therefore the ratio  $L/\mu$  represents the degree of nonlocality  $\alpha$ . In the strongly nonlocal limit,  $\alpha \gg 1$ , we have

$$\gamma \approx \frac{\alpha\sqrt{\pi}}{2\mu^2} \quad (20)$$

It can be seen that the phase shift rate of SNSOS is proportional to the degree of nonlocality, which is at least one order larger than that for local solitons.

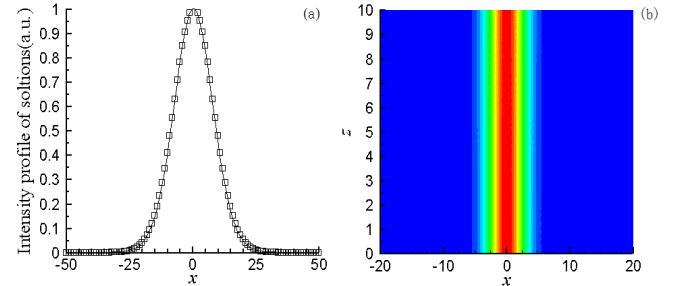


Fig. 1: (a) The intensity profile of the (1+1)D soliton in LG for  $\alpha = 150$ . The squares represent the iterative solution and the solid line represents the perturbative solution expressed in Eq.(17). (b) Simulation of the soliton propagation where the perturbative solution (solid line in (a)) serves as the incident profile.

Fig. 1(a) displays the intensity profile of the (1+1)D SNSOS in LG with  $\alpha = 150$ . In the strongly nonlocal case, the perturbative analytic result (solid line) is very close to the numerical result (squares) denoted by the good agreement between them. Fig. 1(b) shows the simulation of the light beam propagation in the form of soliton with an input amplitude profile described by Eq.(17). The phase shift of the (1+1)D SNSOS in LG versus the propagation distance is manifested in Fig. 2. Under different conditions of nonlocalities, the higher the degree of nonlocality, the faster the phase shift gets. The phase shift rates of SNSOSs are obtained by calculating the slopes of the data in Fig.2. It suggests that

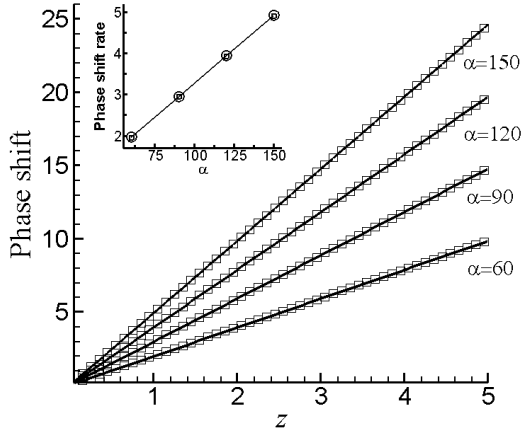


Fig. 2: Comparison of the phase shifts of SNSOS in (1+1)D LG with different degree of nonlocality  $\alpha$  between the perturbative analytic results (solid lines) with the numerical results (squares). Inset shows the phase shift rate as a function of the degree of nonlocality  $\alpha$ . Circles and squares are respectively calculated based on the analytical and numerical results. Solid line is provided as a guide to the eye.

SNSOS experiences  $\pi$  phase shift within one Rayleigh distance in (1+1)D LG. The inset of Fig. 2 reveals the phase shift rate changes along with the degree of the nonlocality. It visualizes that the enhancement-effect of the nonlocality on the phase shift rate acts linearly and effectively. One can, therefore, obtain faster phase shift by directly enlarging the size of the LG since the degree of nonlocality of LG is determined by the glass size.

### 3. Discussion

In the previous section, we investigate the phase shift of SNSOS in (1+1)D LG and find that the phase shift rate is proportional to the degree of nonlocality. A quantitative comparisons of phase shift rates of spatial solitons in (1+1)D and (1+2)D materials with different response functions are represented in Table 1 [17, 20–22, 24]. Fig. 3 gives an illustration of the phase shift rates of solitons versus the degree of nonlocality  $\alpha$  in different local and nonlocal systems. There are several features that should be emphasized. First of all, solitons propagating in material with nonlocal nonlinear response have much faster phase shift rate. We call this ‘nonlocality-enhanced phase shift’. In nonlocal media with Gaussian response functions, nonlocality-enhancing factor is  $\alpha^2$ , which is more than 100, in the strongly nonlocal cases [17]. In LG, the nonlocality-enhancing factors are much smaller but still over 10 for the lower limit of the strong nonlocality [22]. Second, the nonlocality-enhancing factors present different forms in (1+1)D and (1+2)D systems. Generally speaking, compared with higher dimensional SNSOSs, lower dimensional SNSOSs have much faster phase shift rates. Specifically, in (1+1)D NLC and LG, phase shift rates have the same expressions which are proportional to the degree of nonlocality  $\alpha$

Table 1: Phase shift rate of spatial solitons in (1+1)D and (1+2)D materials with different response functions

Dimension	Nonlocality	Material	Phase shift rate
(1+1)D	Local	Local media	$1/L_R$ [24]
	Nonlocal	Media with Gaussian response	$\alpha^2/L_R$ [17]
		NLC	$\sqrt{\pi}\alpha/L_R$ [20]
		LG	$\sqrt{\pi}\alpha/L_R$
(1+2)D	Local	Local media	unstable
	Nonlocal	Media with Gaussian response	$\alpha^2/L_R$ [17]
		NLC	$(8.6\ln\alpha - 4.3)/\pi L_R$ [21]
		LG	$(2\ln\alpha + 6.24)/L_R$ [22]

<sup>1</sup>  $L_R$  is the Rayleigh distance.

<sup>2</sup>  $\alpha$  is the degree of nonlocality defined as the ratio of the characteristic length of the response function to the beam width. In LG, the characteristic length of the response function is the medium size [22].

[20]. This indicates the phase shift rates of SNSOS are more than one order of magnitude faster than those for the local ones. While in the (1+2)D NLC and LG, the phase shift rate takes the form of natural logarithm of  $\alpha$  [21, 22]. Smaller although than that in the lower-dimensional nonlocal media, nonlocality-enhancing factor is still one order larger than the result for local solitons in LG, and more that 5 times larger than the result for local solitons in NLC. The nonlocality-enhancement-effect on the phase shift of SNSOS originates from the fact that, the refractive index, directly contributing to the phase shift, is induced not by the light intensity but, thanks to the ‘averaging-effect’ of the nonlocality, by the power of the whole beam.

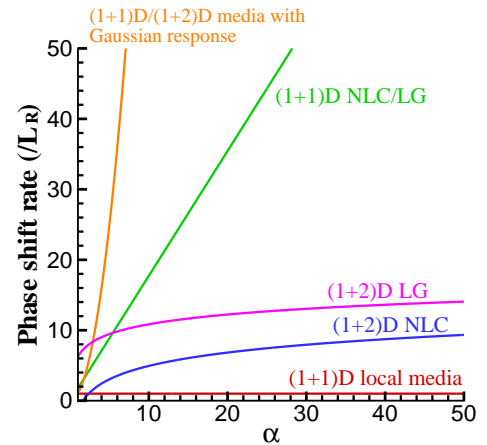


Fig. 3: Comparison of the phase shift rates of solitons versus the degree of nonlocality  $\alpha$  in different local and nonlocal systems. The solid curves are the phase shift rates, normalized by Rayleigh distance  $L_R$  in, from top to bottom, (1+1)D or (1+2)D media with Gaussian response, (1+1)D NLC or LG, (1+2)D LG, (1+2)D NLC, (1+1)D local media, respectively.

#### 4. Conclusion

Using the perturbation method, we investigate the large phase shift of SNSOS in (1+1)D LG. The perturbative solution of the fundamental soliton under the second approximation suggests that the phase shift rate of (1+1)D SNSOS is proportional to the degree of the nonlocality, which is at least one order faster than that of its local counterpart. This facilitates a  $\pi$ -phase-shift within one Rayleigh distance in (1+1)D LG. The nonlocality-enhancement-effect on the phase shift of SNSOS is an important and intrinsic feature of nonlocality, which, although works differently in different nonlocal systems, leads to a much faster, generally one order faster, phase shift rate of SNSOS than that of the local counterpart. Phase shift is very important for modification, manipulation, and control of optical field based on the principle of interference. The nonlocality-enhancement to the phase shift might be of great potential in applications based on the effective generation of large phase shift.

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